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Positive Values of Inhomogeneous Quadratic Forms of Signature -2

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Let $Q(x_1, ..., x_n)$ be a real indefinite quadratic form of type $(r, n-r)$ $(1 \leqslant r \leqslant n-1)$ and determinant $D \neq 0$. We know that there exist constants Γ, independent of Q and dependent only on n and r such that given any reals $r_1, ..., r_n$, there exist integers $x_1, ..., x_n$ such that

$$0 < Q(x_1 + r_1, ..., x_n + r_n) \leqslant (\Gamma\,|D|)^{1/n}.$$

Let $\Gamma_{r,n-r}$ denote the infimum of all such numbers Γ. Then in this we prove that $\Gamma_{r,r+2} = (2^{2r+2})/3$ for $r \geqslant 4$.

1. INTRODUCTION

Let $Q(x_1, ..., x_n)$ be a real indefinite quadratic form of type $(r, n-r)$ $(1 \leqslant r \leqslant n-1)$ and determinant $D \neq 0$. Blaney [4] has shown that there exist constants Γ, independent of Q and dependent only on n and r such that given any reals $r_1, ..., r_n$ there exist $(x_1, ..., x_n) \equiv (r_1, ..., r_n) \pmod 1$ satisfying

$$0 < Q(x_1, ..., x_n) \leqslant (\Gamma\,|D|)^{1/n}. \tag{1.1}$$

Let $\Gamma_{r,n-r}$ denote the infimum of all such numbers Γ. Bambah, Dumir, and Hans-Gill have shown that

(1) $\Gamma_{r+3,r} = 2^{2r+1}$ $(r \geqslant 1)$ [1],
(2) $\Gamma_{r+2,r} = (2^{2r+2})/3$ $(r \geqslant 1)$ [1],
(3) $\Gamma_{r+1,r} = 2^{2r}$ $(r \geqslant 1)$ [1],
(4) $\Gamma_{r,r} = 2^{2r}$ $(r \geqslant 1)$ [2],
(5) $\Gamma_{r,r+1} = 2^{2r}$ $(r \geqslant 3)$ and $\Gamma_{2,3} = (\frac{7}{4})^5$ [3].

Earlier Dumir had also proved $\Gamma_{1,2} = 8$ [6]. Thus minima of positive values of inhomogeneous quadratic forms of signatures 0, ± 1, 2, and 3 are

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known completely. In this paper we shall consider the minima of positive values of signature -2.

Dumir and Hans-Gill [8] have already proved that $\Gamma_{1,3} = 16$. Also from a result of Jackson [10] and Theorem 3(a) of Bambah, Dumir, and Hans-Gill [2], it follows that $\Gamma_{2,4} \leqslant 64$. In this paper we shall take up the quadratic forms of type $(r, r+2)$ $(r \geqslant 3)$. To be precise, we shall prove the following results:

THEOREM 1. *Let $Q_n(x_1, ..., x_n)$ $(n \geqslant 8)$ be a n-ary zero quadratic form of signature -2 and determinant $D \neq 0$. Then given any real numbers $r_1, ..., r_n$ there exist integers $x_1, ..., x_n$ such that*

$$0 < Q_n(x_1 + r_1, ..., x_n + r_n) \leqslant \left(\frac{2^n}{3} |D|\right)^{1/n} = K. \tag{1.2}$$

This result is the best possible.

THEOREM 2. *Let $Q_8 = Q_8(x_1, ..., x_8)$ be a non-zero quadratic form of type $(3, 5)$ and* $\det D \neq 0$. *Then given any real numbers $r_1, ..., r_n$ there exist integers $x_1, ..., x_n$ such that*

$$0 < Q_8(x_1 + r_1, ..., x_8 + r_8) < (116 |D|)^{1/8}.$$

[The constant 116 can be improved. However, Theorem 2 is good enough to prove the result for $n \geqslant 10$]. Combining Theorems 1 and 2, we get

THEOREM 3. $\Gamma_{3,5} < 116$.

For $r \geqslant 4$, we prove

THEOREM 4. $\Gamma_{r,r+2} = (2^{2r+2})/3$ $(r \geqslant 4)$.

2. Zero Forms

First we shall deal with zero forms by giving a few definitions.

DEFINITION 2.1. Let α, β be any given real numbers and let $Q(x_1, ..., x_n)$ be any quadratic form. We say

$$\alpha < Q(x_1, ..., x_n) \leqslant \beta \tag{2.1}$$

is solvable if for any reals $r_1, ..., r_n$, there exist $(x_1, ..., x_n) \equiv (r_1, ..., r_n)$ (mod 1) satisfying (2.1).

DEFINITION 2.2. Let $a \neq 0$ be any real number. We say a quadratic form represents the number "a" if there exist integers $x_1, ..., x_n$ such that $Q(x_1, ..., x_n) = a$.

Before starting with the proof of Theorem 1, we state a few results without proofs.

LEMMA 2.1. *Let α, β, x_0, and λ be real numbers with $\lambda > 1$. Let m be an integer given by $m < \lambda \leqslant m+1$. Then*

(a) *there exists $x \equiv x_0 \pmod 1$ satisfying*

$$0 < (x+\alpha)^2 + \beta \leqslant \lambda$$

provided $-m^2/4 < \beta \leqslant \lambda - \frac{1}{4}$.

(b) *There exists $x \equiv x_0 \pmod 1$ satisfying*

$$0 < -(x-\alpha)^2 + \beta \leqslant \lambda$$

provided $\frac{1}{4} < \beta \leqslant m^2/4 + \lambda$.

For the proof of (a), see Lemma 6 of Dumir [7] and for the proof of (b) see Lemma 2 of Dumir [6].

LEMMA 2.2. *Let $Q(x_1, ..., x_n)$ be a real indefinite zero quadratic form of determinant $D \neq 0$. Then given reals α and β,*

$$\alpha < Q(x_1, ..., x_n) \leqslant \beta$$

is solvable provided $(\beta - \alpha) \geqslant 2(|D|)^{1/n}$.

For the proof, see Jackson [10].

LEMMA 2.3. *Let $Q(x, y)$ be a positive definite quadratic form of determinant Δ. Then there exist integers x, y such that*

$$0 < Q(x, y) \leqslant (\tfrac{4}{3} |\Delta|)^{1/2}.$$

It is the classical result of Lagrange (1773).

LEMMA 2.4. *Let t and A be positive numbers and let β, λ be any real numbers. Let $2h$, k be integers such that*

$$\left| ht - \frac{k^2}{t^2} \right| + \frac{t}{2} \leqslant A.$$

Suppose that either $t^3 \neq k^2/h$ *or* $\beta \not\equiv th/k \pmod{t/k, 2/t^2}$ (*i.e.*, $\beta - th/k$ *is not an integral combination of* t/k *and* $2/t^2$). *Then there exist integers* x, y *satisfying*

$$0 < tx + \beta y - \frac{y^2}{t^2} + \lambda \leqslant A.$$

For the proof, see Lemma 6 of Macbeath [11].

LEMMA 2.5. *Let* $Q(x, y, z, t)$ *be a non-zero quaternary quadratic form of type* (3, 1) *and determinant* $\Delta \neq 0$. *Then there exist integers* x, y, z, t; *not all zero, such that*

$$0 < |Q(x, y, z, t)| \leqslant (\tfrac{2}{9} |\Delta|)^{1/4}$$

except when Q *is equivalent to a positive multiple of the following*:

$$G_1 = x^2 + y^2 + z^2 - t^2 - xt - yt - zt$$

or

$$G_2 = x^2 + xy - y^2 + 2(z^2 + zt + t^2)$$

or

$$G_3 = x^2 + xy + y^2 + 2(z^2 + zt - t^2).$$

Moreover, for each $i = 1, 2, 3$, *there exist integers* x, y, z, t, *not all zero, satisfying*

$$0 < |G_i(x, y, z, t)| \leqslant (\tfrac{4}{7} |\Delta|)^{1/4}.$$

For the proof, see Oppenheim [12].

Remark 2.1. We know by the results of Oppenheim ([13, 11]) that a zero incommensurable form in $n \geqslant 4$ variables takes arbitrarily small non-zero values of either sign. Also by Watson [15], for such forms, given any reals α, δ ($\delta > 0$),

$$\alpha < Q(x_1, ..., x_n) < \alpha + \delta$$

is solvable. So by taking $\alpha = 0$, $\delta < \Gamma_{r,n-r}$ results hold in this case. Thus we need consider only zero rational forms.

LEMMA 2.6. *Let* $Q_n = Q_n(x_1, ..., x_n)$ ($n \geqslant 8$) *be a rational form of signature* -2 *and determinant* $D \neq 0$. *Then* (1.2) *is solvable if* Q_n *represents a number "a" with* $0 < |a| < K/2$.

Proof. Without loss of generality, we can suppose that representation of a by Q_n is primitive. Replacing Q_n by an equivalent form, if necessary, we can suppose that

$$Q_n = a(x_1 + h_2 x_2 + \cdots + h_n x_n)^2 + Q_{n-1}(x_2, ..., x_n).$$

By homogeneity, we can suppose that $|a| = 1$ so that $K > 2$. Also Q_{n-1} is a rational form and so by Meyer's Theorem, Q_{n-1} is also a zero form since $n \geqslant 8$. Now we distinguish the two cases.

Case I. $a = 1$. Then

$$Q_n = (x_1 + h_2 x_2 + \cdots + h_n x_n)^2 + Q_{n-1}(x_2, ..., x_n).$$

Let m be an integer given by $m < K \leqslant m+1$. Then $m \geqslant 2$. Also by Lemma 2.1(a), (1.2) is solvable if

$$-\frac{m^2}{4} < Q_{n-1}(x_2, ..., x_n) \leqslant K - \frac{1}{4}. \tag{2.2}$$

Since Q_{n-1} is a zero form, so by Lemma 2.2, (2.2) is solvable if

$$K - 1/4 + m^2/4 \geqslant 2(|D|)^{1/n-1} = (3/2)^{1/n-1} K^{n/n-1}. \tag{2.3}$$

Since for any fixed m,

$$f(K) = K^{-1/n-1} + \frac{m^2-1}{4} K^{-n/n-1}$$

is a decreasing function of K for $K > 1$ and $K \leqslant m+1$, so $f(K) \geqslant f(m+1) = [(m+3)/4](m+1)^{-1/n-1} = g(m)$ where $g(m)$ is an increasing function of m and so

$$f(K) \geqslant g(2) = \tfrac{5}{4} \cdot 3^{-1/n-1} > (\tfrac{3}{2})^{1/n-1} \qquad (\text{since } n \geqslant 8).$$

Case II. $a = -1$. Then

$$Q_n = -(x_1 + h_2 x_2 + \cdots + h_n x_n)^2 + Q_{n-1}(x_2, ..., x_n).$$

By Lemma 2.1(b), (1.2) is solvable if

$$\frac{1}{4} < Q_{n-1} \leqslant K + \frac{m^2}{4}$$

is solvable. Further by Lemma 2.2, it is so if

$$K + \frac{m^2}{4} - \frac{1}{4} \geqslant 2(|D|)^{1/n-1}$$

which is the same as (2.3) and hence is satisfied. Thus the lemma is proved completely.

3. Reduction

First we shall prove Theorem 1 for $n=8$ so that we shall prove the following result.

THEOREM A. *Let $Q_8 = Q_8(x_1, ..., x_8)$ be a rational quadratic form of the type* (3, 5) *and determinant $D \neq 0$. Let $K = 2(|D|/3)^{1/8}$. Then*

$$0 < Q_8(x_1, ..., x_8) \leqslant K \tag{3.1}$$

is solvable.

Remark 3.1. First we shall reduce Q_8 to a well-behaved quadratic form. Since Q_8 is a rational form in eight variables, by Meyer's Theorem, Q_8 is a zero form. By using arguments similar to Birch [5] and by taking a rational multiple, if necessary, we assume

$$\begin{aligned} Q_8 = {} & (x_1 + a_2 x_2 + \cdots + a_8 x_8)\, x_2 + m(x_3 + b_4 x_4 + \cdots + b_8 x_8)\, x_4 \\ & + Q_4(x_5, x_6, x_7, x_8), \end{aligned} \tag{3.2}$$

where m is a natural number and Q_4 is a rational *quadratic* form of type (1, 3). Further if Q_4 is a zero form, we can write (3.2) as

$$\begin{aligned} Q_8 = {} & (x_1 + a_2 x_2 + \cdots + a_8 x_8)\, x_2 + m_1(x_3 + b_4 x_4 + \cdots + b_8 x_8)\, x_4 \\ & + m_2(x_5 + c_6 x_6 + c_7 x_7 + c_8 x_8)\, x_6 - (\alpha x_7^2 + \beta x_7 x_8 + \gamma x_8^2), \end{aligned} \tag{3.3}$$

where m_1, m_2 are natural numbers and $Q_2 = \alpha x_7^2 + \beta x_7 x_8 + \gamma x_8^2$ is a reduced positive definite form, i.e.,

$$0 \leqslant \beta \leqslant \alpha \leqslant \gamma. \tag{3.4}$$

Without loss of generality, we assume that in either case

$$-\tfrac{1}{2} < a_i \leqslant \tfrac{1}{2}, \qquad -\tfrac{1}{2} < b_j \leqslant \tfrac{1}{2} \tag{3.5}$$

for all i, j. And in case Q_8 is in the form (3.3) we also have

$$-\tfrac{1}{2} < c_k \leqslant \tfrac{1}{2} \qquad \text{for all } k. \tag{3.6}$$

Further by Watson [16, p. 21], we may assume that if $a_2 = 0$, all a's are zero, if $b_4 = 0$, all b's are zero, and if $c_6 = 0$, all c's are zero. We further assume

$$-\tfrac{1}{2} < r_i \leqslant \tfrac{1}{2} \qquad \text{for} \quad i = 1, ..., 8. \tag{3.7}$$

Now we shall prove Theorem A by means of a number of lemmas.

LEMMA 3.1. *Theorem A holds if $K \geqslant 1$.*

Proof. Choose $(x_3, ..., x_8) \equiv (r_3, ..., r_8) \pmod 1$ arbitrarily. Choose $x_2 \equiv r_2 \pmod 1$ such that $0 < x_2 \leqslant 1$ and $x_1 \equiv r_1 \pmod 1$ such that

$$0 < Q_8 = x_1 x_2 + \mu \leqslant x_2 \leqslant 1 \leqslant K,$$

where μ is some constant. This proves the lemma.

Remark 3.2. The same proof is valid for general n if $K = ((2^n/3)\,|D|)^{1/n} \geqslant 1$.

In view of the above lemma, henceforth we assume $K < 1$.

LEMMA 3.2. *Theorem A is true if any of the m's in* (3.2) *or* (3.3) *is not equal to* 1.

Proof. (i) Let Q_8 be in the form (3.2), so that Q_4 is a non-zero form of type (1, 3). If possible, let $m \neq 1$. Then $m \geqslant 2$. By Lemma 2.5, $-Q_4$ and hence Q_8 represents a number "a" such that

$$0 < a^4 \leqslant \tfrac{4}{7}\,|\Delta|,$$

where Δ is the determinant of the form Q_4. Also

$$\begin{aligned} K^8 &= \frac{2^8}{3}\,|D| = \frac{2^8}{3} \cdot \frac{1}{4} \cdot \frac{m^2}{4} \cdot |\Delta| \geqslant \frac{2^8}{3} \cdot \frac{1}{4} \cdot \frac{4}{4} \cdot \frac{7a^4}{4} \\ &= \frac{112}{3}\,a^4 \qquad \therefore a^4 \leqslant \frac{3}{112}\,K^8 < \frac{K^4}{16} \;(\because K < 1). \end{aligned}$$

So $|a| < K/2$.

(ii) Now let Q_8 be in the form (3.3). Then $Q_2 = \alpha x_7^2 + \beta x_7 x_8 + \gamma x_8^2$ is a positive definite form and so by Lemma 2.3, Q_2 and hence Q_8 represents a number "a" such that

$$0 < a^2 \leqslant \tfrac{4}{3}\,|\Delta| \qquad \text{where} \quad \Delta = \det Q_2.$$

Let at least one of m_1 or m_2 not be equal to 1. Then

$$\begin{aligned} K^8 &= \frac{2^8}{3} \cdot |D| = \frac{2^8}{3} \cdot \frac{1}{4} \cdot \frac{m_1^2}{4} \cdot \frac{m_2^2}{4}\,|\Delta| \\ &\geqslant \frac{2^8}{3} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{4}{4} \cdot \frac{3}{4} \cdot a^2 \qquad \therefore a^2 \leqslant \frac{K^8}{4} < \frac{K^2}{4} \;(\because K < 1) \end{aligned}$$

and so $|a| < K/2$.

Thus in both cases, Q_8 represents a number "a" with $|a| < K/2$ and so by Lemma 2.6, Theorem A holds.

Remark 3.3. From now on, we shall assume that all the m's are equal to 1, so that if Q_4 is a non-zero form, then

$$\begin{aligned} Q_8 = {} & (x_1 + a_2 x_2 + \cdots + a_8 x_8)\, x_2 + (x_3 + b_4 x_4 + \cdots + b_8 x_8)\, x_4 \\ & + Q_4(x_5, x_6, x_7, x_8) \end{aligned} \tag{3.8}$$

and if Q_4 is a zero form, then

$$\begin{aligned} Q_8 = {} & (x_1 + a_2 x_2 + \cdots + a_8 x_8)\, x_2 \\ & + (x_3 + b_4 x_4 + \cdots + b_8 x_8)\, x_4 + (x_5 + c_6 x_6 + \cdots + c_8 x_8)\, x_6 \\ & - (\alpha x_7^2 + \beta x_7 x_8 + \gamma x_8^2) \end{aligned} \tag{3.9}$$

and satisfies all the conditions stipulated in Remark 3.1. In particular the conditions (3.4)–(3.7) are satisfied.

LEMMA 3.3. *Theorem A is true if* $K \leqslant 0.7$.

Proof. By using arguments similar to the ones used in the last lemma, if Q_4 is a zero form, then Q_8 represents a number "a" such that

$$0 < a^2 \leqslant K^8 = K^6 \cdot K^2 < \frac{K^2}{4},$$

and if Q_4 is a non-zero form, then Q_8 represents a number "a" such that

$$0 < a^4 \leqslant \frac{3}{28} K^8 < \frac{K^4}{16}.$$

Thus in both cases, $|a| < K/2$ and so by Lemma 2.6, Theorem A holds.

Remark 3.4. In view of the Remark 3.2 and the last lemma, we shall assume

$$0.7 < K < 1. \tag{3.10}$$

LEMMA 3.4. *Theorem A holds if* $r_2 \neq 0$.

Proof. Choose $(x_3, \ldots, x_8) \equiv (r_3, \ldots, r_8) \pmod 1$ arbitrarily and $x_2 = r_2$ so that $0 < |x_2| \leqslant \frac{1}{2}$. Then

$$Q_8 = x_1 x_2 + \mu,$$

where μ is some constant. Now choose $x_1 \equiv r_1 \pmod 1$ such that

$$0 < Q_8 = x_1 x_2 + \mu \leqslant |x_2| \leqslant \tfrac{1}{2} < K.$$

Remark 3.5. The same proof is valid for general n if $K = (2^n/3\,|D|)^{1/n} \geqslant \frac{1}{2}$. Also in a similar way, we can prove Theorem A holds if $r_4 \neq 0$. Further if Q_4 is a zero form, then similarly Theorem A holds if $r_6 \neq 0$. So we assume $r_2 = r_4 = 0$ and $r_6 = 0$ if Q_4 is a zero form.

Remark 3.6. Lemma 3.4 holds even if $K \geqslant \frac{1}{2}$.

Now we deal with the cases when Q_4 is a zero form and when Q_4 is a non-zero form separately.

4. Q_4 Is a Non-zero Form

LEMMA 4.1. *Theorem A is true if* $Q_4 \not\sim -\rho G_i$ $(i = 1, 2, 3)$ *and* $\rho > 0$.

Proof. Since $Q_4 \not\sim -\rho G_i$ $(i = 1, 2, 3)$ and $\rho > 0$, by Lemma 2.5, $-Q_4$ and hence Q_8 represents a number "a" such that

$$0 < a^4 \leqslant \tfrac{2}{9}\,|\Delta|.$$

Now

$$K^8 = \frac{2^8}{3}\,|D| = \frac{2^8}{3}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot|\Delta| \geqslant \frac{2^8}{3}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{9}{2}\,a^4 = 24a^4$$

$$a^4 \leqslant \frac{K^8}{24} < \frac{K^4}{24} \Rightarrow |a| < \frac{K}{2} \qquad \text{as} \quad K < 1.$$

Theorem A follows from Lemma 2.6.

LEMMA 4.2. *Theorem A holds if* $Q_4 \sim -\rho G_2$ *or* $Q_4 \sim -\rho G_3$, $\rho > 0$.

Proof. We note $|\det G_2| = \frac{15}{4} = |\det G_3|$

$$K^8 = \frac{2^8}{3}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{15}{4}\,\rho^4 = 20\rho^4$$

$$\rho^4 = \frac{K^8}{20} < \frac{K^4}{16} \Rightarrow \rho < \frac{K}{2}.$$

But Q_8 represents $-\rho$ and so Theorem A is true by Lemma 2.6.

LEMMA 4.3. *Theorem* A *is true if* $Q_4 \sim -\rho G_1$, *where* $\rho > 0$ *and* $\rho \neq \frac{1}{2}$.

Proof. Without loss of generality, we assume $Q_4 = -\rho G_1$. Clearly the minimum value attained by $|Q_4|$ is ρ and $\rho = (\frac{4}{7}|\Delta|)^{1/4}$. Also Q_8 represents $-\rho$.

If $0 < \rho < K/2$, we are clearly through by Lemma 2.6. So assume

$$\frac{K}{2} \leqslant \rho = \left(\frac{4}{7}|\Delta|\right)^{1/4} = \left(\frac{4}{7}\cdot 2^4\,|D|\right)^{1/4} = \left(\frac{3}{28}K^8\right)^{1/4}. \tag{4.1}$$

Therefore

$$\frac{1}{16}K^4 \leqslant \frac{3}{28}K^8 \qquad \text{or} \qquad K^4 \geqslant \frac{7}{12}, \qquad \text{i.e.,} \quad K \geqslant \left(\frac{7}{12}\right)^{1/4}.$$

$$\frac{1}{2}\left(\frac{7}{12}\right)^{1/4} \leqslant \rho \leqslant \left(\frac{3}{28}K^8\right)^{1/4} < \left(\frac{3}{28}\right)^{1/4} \qquad (\because K < 1).$$

Let $t^3 = 1/\rho$. Then $(\frac{28}{3})^{1/4} < t^3 \leqslant 2\cdot(\frac{12}{7})^{1/4}$. Now we want to prove that there exist $(x_1, ..., x_8) \equiv (r_1, ..., r_8) \pmod 1$ such that

$$\begin{aligned} 0 < (x_1 + a_2x_2 + \cdots + a_8x_8)\,x_2 + (x_3 + b_4x_4 + \cdots + b_8x_8)\,x_4 \\ +\ \rho(x_5^2 - x_6^2 - x_7^2 - x_8^2 + x_5x_6 + x_5x_7 + x_5x_8) < K \end{aligned} \tag{4.2}$$

is solvable. Since $r_2 = r_4 = 0$, we take $x_2 = 1$, $x_4 = 0$, and $(x_3, x_5, x_6x_7) \equiv (r_3, r_5, r_6, r_7) \pmod 1$ arbitrarily. Also take $x_1 = x + r_1$, $x_8 = y + r_8$ so that (4.2) is satisfied if there exist integers x, y such that $0 < x + a_8(y + r_8) - \rho(y + r_8)^2 + \rho x_5(y + r_8) + \lambda \leqslant K$, where λ is a constant or

$$0 < x + (a_8 - 2\rho r_8 + \rho x_5)\,y - \rho y^2 + \mu \leqslant K, \tag{4.3}$$

where μ is some constant. Since $t^3 = 1/\rho$, (4.3) is satisfied if

$$0 < tx + t(a_8 - 2\rho r_8 + \rho x_5)\,y - \frac{1}{t^2}\,y^2 + t\mu \leqslant tK. \tag{4.4}$$

Since $\rho \neq \frac{1}{2}$, $t^3 \neq 2$ so that by Lemma 2.4 with $h = \frac{1}{2}$, $k = 1$, (4.4) is satisfied if

$$\left|\frac{t}{2} - \frac{1}{t^2}\right| + \frac{t}{2} \leqslant tK \tag{4.5}$$

or

$$\left|\frac{1}{2} - \rho\right| + \frac{1}{2} \leqslant K. \tag{4.6}$$

If $\rho > \frac{1}{2}$, (4.6) is satisfied if

$$\rho \leqslant K,$$

which is so since by (4.1),

$$\rho = \left(\frac{3}{28} K^8\right)^{1/4} < K^2 < K \qquad (\text{since } K < 1).$$

If $\rho < \frac{1}{2}$, (4.5) is satisfied if $\rho + K \geqslant 1$ which is satisfied since $\rho \geqslant K/2$ and $K > 0.7$. Hence the lemma follows.

LEMMA 4.4. *Theorem* A *is true if* $Q_4 = -\frac{1}{2} G_1$.

Proof. We proceed as in the last lemma. Now we set $\rho = \frac{1}{2}$ in (4.4) and we get that we are through if there exist integers x, y satisfying

$$0 < tx + t\left(a_8 - r_8 + \frac{1}{2} x_5\right) y - \frac{y^2}{t^2} + t\mu \leqslant tK, \tag{4.7}$$

where $t^3 = 1/\rho = 2$. Also (4.5) is clearly satisfied. So by Lemma 2.4 with $h = \frac{1}{2}$, $k = 1$, (4.7) is satisfied unless

$$a_8 - r_8 + \frac{1}{2} x_5 \equiv \frac{1}{2} \pmod 1 \tag{4.8}$$

for all $x_5 \equiv r_5 \pmod 1$. But

$$a_8 - r_8 + \frac{1}{2} r_5 \equiv \frac{1}{2} \pmod 1$$

and

$$a_8 - r_8 + \frac{1}{2}(r_5 + 1) \equiv \frac{1}{2} \pmod 1$$

cannot hold simultaneously so that (4.8) does not hold either for $x_5 = r_5$ or $x_5 = r_5 + 1$. So (4.7) is satisfied for some integers x, y and this proves the lemma.

Remark 4.1. Combining Lemmas 4.1–4.4, we get that Theorem A is true if Q_4 is a non-zero form.

5. Q_4 IS A ZERO FORM

Let Q_4 be a zero form and Q_8 be in the form (3.9). Write $Q_2(x, y) = \alpha x^2 + \beta xy + \gamma y^2$. Then $0 \leqslant \beta \leqslant \alpha \leqslant \gamma$. Let $\Delta = \det Q_2 = \alpha\gamma - \beta^2/4$. Also Q_2 is a positive definite reduced form and so

$$0 < \alpha \leqslant (\tfrac{4}{3} |\Delta|)^{1/2}. \tag{5.1}$$

Now we prove a few results.

LEMMA 5.1. *Theorem* A *is true if* $\Delta \geqslant \frac{3}{4}$.

Proof.

$$K^8 = \frac{2^8}{3}\,|D| = \frac{2^8}{3}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot|\Delta|$$

$$\geqslant \frac{2^8}{3}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4} = 1$$

and so $K \geqslant 1$ and Theorem A follows from Lemma 3.1.

Remark 5.1. In view of the above lemma, we assume $\Delta < \frac{3}{4}$. Then from (5.1), we get $\alpha < 1$.

LEMMA 5.2. *Theorem* A *is true if* $\alpha \neq \frac{1}{2}$ *or if* $\beta \neq 0$.

Proof. We note

$$K^8 = \frac{2^8}{3}\cdot|D| = \frac{2^8}{3}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot|\Delta| = \frac{4}{3}\cdot|\Delta|$$

and so $|\Delta| = \frac{3}{4}K^8$.

Since Q_8 represents $-\alpha$, by Lemma 2.6, Theorem A is true if $\alpha < K/2$. So we assume $K/2 \leqslant \alpha$. Thus

$$\frac{K}{2} \leqslant \alpha \leqslant \left(\frac{4}{3}\,|\Delta|\right)^{1/2} = K^4. \tag{5.2}$$

Now set $\alpha = 1/t^3$. Now to prove Theorem A, we want to choose $(x_1, ..., x_8) \equiv (r_1, ..., r_8) \pmod 1$ to satisfy (3.1). By Remark 3.5, $r_2 = r_4 = r_6 = 0$. Choose $x_2 = x_4 = 0$, $x_6 = 1$, $x_5 = x + r_5$, $x_7 = y + r_7$, and $x_8 \equiv r_8 \pmod 1$ (to be specified later). Then we want to prove that there exist integers x, y satisfying

$$0 < x + r_5 + c_6 + c_7(y + r_7) + c_8x_8 - \frac{1}{t^3}\,(y + r_7)^2$$
$$- \beta x_8(y + r_7) - x_8^2 \leqslant K$$

or

$$0 < tx + t\left(c_7 - \frac{2}{t^3}\,r_7 - \beta x_8\right)y - \frac{1}{t^2}\,y^2 + t\mu \leqslant tK, \tag{5.3}$$

where μ is some constant.

Taking $h=\frac{1}{2}$, $k=1$ in Lemma 2.4 (since $\alpha\neq\frac{1}{2}$, $t^3\neq 2$), we see that we are through if

$$\left|\frac{t}{2}-\frac{1}{t^2}\right|+\frac{t}{2}\leqslant tK$$

or

$$\left|\frac{1}{2}-\alpha\right|+\frac{1}{2}\leqslant K. \tag{5.4}$$

(i) If $\alpha<\frac{1}{2}$, (5.4) is true if $1-\alpha\leqslant K$, which is true since $\alpha\geqslant K/2$ and $K>0.7$.

(ii) If $\alpha>\frac{1}{2}$, (5.4) is true if $\alpha-\frac{1}{2}+\frac{1}{2}\leqslant K$ or if $\alpha\leqslant K$ which is so since by (5.2), $\alpha\leqslant K^4<K$ ($\because K<1$). Let us assume now that $\alpha=\frac{1}{2}$ so that $t^3=2$. If $\beta\neq 0$, then $0<\beta\leqslant\alpha=\frac{1}{2}$. Then for $x_8=r_8$ or $x_8=r_8+1$, we will have

$$c_7-r_7-\beta x_8 \not\equiv \tfrac{1}{2} \pmod 1. \tag{5.5}$$

So with this choice of x_8, by Lemma 2.4 with $h=\frac{1}{2}$, $k=1$, (5.3) will be satisfied. So Theorem A holds unless $\alpha=\frac{1}{2}$ and $\beta=0$. This proves the lemma.

Remark 5.2. From now on, we assume $\alpha=\frac{1}{2}$ and $\beta=0$. Then from (5.5), we see we are done if

$$c_7-r_7\neq\tfrac{1}{2} \pmod 1.$$

So we assume

$$c_7-r_7\equiv\tfrac{1}{2} \pmod 1. \tag{5.6}$$

Had we taken $x_6=-1$, similarly we would have gotten

$$-c_7-r_7\equiv\tfrac{1}{2} \pmod 1. \tag{5.7}$$

From (5.6) and (5.7) we get $2c_7\equiv 0 \pmod 1$ or $c_7=0$ or $\frac{1}{2}$. Further

$$c_7=0,\quad r_7=\tfrac{1}{2}\qquad\text{and}\qquad c_7=\tfrac{1}{2},\quad r_7=0, \tag{5.8}$$

which we assume from now on.

Now we are done unless

$$\frac{3}{4}>\Delta=\alpha\gamma-\frac{\beta^2}{4}=\frac{1}{2}\gamma\qquad\text{so that}\quad \gamma<\frac{3}{2}. \tag{5.9}$$

Also

$$K^8 = \frac{2^8}{3} \cdot |D| = \frac{2^8}{3} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot |\Delta|$$

$$= \frac{2^8}{3} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \gamma$$

$$\geqslant \frac{2^8}{3} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{3} \qquad \left(\text{since } r \geqslant \alpha = \frac{1}{2}\right)$$

and so $K > 0.8$.

LEMMA 5.3. *Theorem* A *is true unless* $\gamma = \frac{1}{2}$ *or* $\gamma = 1$.

Proof. Proceeding as in Lemma 5.2, taking $\beta = 0$ we see that we are through if there integers x, y satisfying

$$0 < tx + t(c_8 - 2\gamma r_8)\, y - \frac{1}{t^2}\, y^2 + t\mu \leqslant tK, \tag{5.10}$$

where $t^3 = 1/\gamma$ and μ is a constant. Also $\gamma \geqslant \alpha = \frac{1}{2}$ and so $t^3 = 1/\gamma \leqslant 2$. If $t^3 \neq 2$ so that $t^3 < 2$, then we are through by Lemma 2.4 with $h = \frac{1}{2}$, $k = 1$ if

$$\left|\frac{t}{2} - \frac{1}{t^2}\right| + \frac{t}{2} \leqslant tK, \qquad \text{i.e.,} \quad \left|\frac{1}{2} - \gamma\right| + \frac{1}{2} \leqslant K;$$

i.e., if $\gamma \leqslant K$. So assume $\gamma > K$ and $\frac{1}{2} < \gamma < \frac{3}{2}$.

Taking $h = k = 1$ in Lemma 2.4, we see that we are through if $t^3 \neq 1$ and

$$\left|t - \frac{1}{t^2}\right| + \frac{t}{2} \leqslant tK$$

or if

$$|1 - \gamma| + \tfrac{1}{2} \leqslant K. \tag{5.11}$$

(i) If $\gamma < 1$, (5.11) is satisfied if $1 - \gamma + \frac{1}{2} \leqslant K$ or if $\frac{3}{2} \leqslant K + \gamma$ or if $\frac{3}{2} \leqslant 2K$ (since $\gamma < K$) or if $\frac{3}{4} \leqslant K$, which is so since $K > 0.8$.

(ii) If $\gamma > 1$, (5.11) is satisfied if

$$\gamma - 1 + \tfrac{1}{2} \leqslant K$$

or if

$$\gamma \leqslant K + \tfrac{1}{2}.$$

So we consider now $K + \frac{1}{2} < \gamma < \frac{3}{2}$.

Since $t^3 = 1/\gamma \neq \frac{2}{3}$, by taking $h = \frac{3}{2}$, $k = 1$ in Lemma 2.4, (5.10) will be satisfied if

$$\left|\frac{3}{2}t - \frac{1}{t^2}\right| + \frac{t}{2} \leqslant tK$$

or if

$$\left|\frac{3}{2} - \gamma\right| + \frac{1}{2} \leqslant K$$

or if $\frac{3}{2} - \gamma + \frac{1}{2} \leqslant K$ or if $2 \leqslant \gamma + K$ or if $2 < K + \frac{1}{2} + K$ ($\because \gamma > K + \frac{1}{2}$) or if $\frac{3}{2} < 2K$ or if $\frac{3}{4} < K$, which is so since $K > 0.8$. Thus the theorem is true unless $t^3 = 2$ or $t^3 = 1$, i.e., unless $\gamma = \frac{1}{2}$ or $\gamma = 1$.

Remark 5.3. From now on, we assume $\gamma = \frac{1}{2}$ or $\gamma = 1$. If $\gamma = \frac{1}{2}$, as in Remark 5.2, we see that Theorem A is true unless $c_8 = 0$ and $r_8 = \frac{1}{2}$ or $c_8 = \frac{1}{2}$ and $r_8 = 0$. If $\gamma = 1$, we see that Theorem A is true unless

$$c_8 - 2r_8 \equiv 0 \pmod{1}. \tag{5.12}$$

As in Remark 5.2, we get that Theorem A is true unless

$$-c_8 - 2r_8 \equiv 0 \pmod{1}.$$

So $2c_8 \equiv 0 \pmod{1}$, i.e., $c_8 = 0$ or $\frac{1}{2}$. If $c_8 = 0$, from (5.12) we get $r_8 = 0$ or $\frac{1}{2}$. If $c_8 = \frac{1}{2}$, we get $2r_8 = c_8 \equiv \frac{1}{2} \pmod{1}$ and so $r_8 = \frac{1}{4}$ or $-\frac{1}{4}$.

LEMMA 5.4. *Theorem* A *is true if* $0 < |a_2| < \frac{1}{2}$.

Proof. If $0 < |a_2| \leqslant 0.4$, take $x_2 = 1$ and $x_i = 0$ for $i \neq 2$. Then $Q_8 = a_2$ so that $0 < |Q_8| \leqslant 0.4 < K/2$.

If $0.4 < |a_2| < 0.5$, take $x_1 = 1$, $x_2 = 2$ or -2 such that $(a_2 x_2) < 0$ and $x_i = 0$ for $i \geqslant 3$. Then $0 < |Q_8| = 2 - 4|a_2| \leqslant 0.4 < K/2$. In both cases, the result follows from Lemma 2.6.

Remark 5.4. In view of above lemma and condition (5.5), we assume $a_2 = 0$ or $\frac{1}{2}$. In a similar way, we can prove that we can take $b_4 = 0$ or $\frac{1}{2}$ and $c_6 = 0$ or $\frac{1}{2}$, otherwise Theorem A is true.

LEMMA 5.5. *Theorem* A *is true if* $0 < |a_3| < \frac{1}{2}$.

Proof. Since $a_3 \neq 0$; by Remark 3.1, $a_2 \neq 0$ and then by the above remark, $a_2 = \frac{1}{2}$. Now if $0 < |a_3| \leqslant 0.1$, take $x_1 = -1$, $x_2 = 2 = x_3$, and $x_i = 0$ for all $i \geqslant 4$. Then $Q_8 = 4a_3$ and $0 < |Q_8| \leqslant 0.4$. If $0.1 < |a_3| < 0.5$, take $x_1 \equiv 0$, $x_2 = 1$, $x_3 = 1$ or -1 such that $(x_3 a_3) < 0$ and $x_i = 0$ for $i \geqslant 4$. Then $Q_8 = \frac{1}{2} - |a_3|$ and $0 < Q_8 < 0.4$. Again as in the above lemma, Theorem A is true.

Remark 5.5. In view of the above lemma and condition (3.5), we assume $a_3=0$ or $\frac{1}{2}$. In a similar way, we can prove $b_5=0$ or $\frac{1}{2}$ as otherwise Theorem A is true.

LEMMA 5.6. *Theorem* A *is true if* $\gamma=\frac{1}{2}$.

Proof. Here

$$Q_8=(x_1+a_2x_2+\cdots+a_8x_8)\,x_2+(x_3+b_4x_4+\cdots+b_8x_8)\,x_4$$
$$+\,(x_5+c_6x_6+c_7x_7+c_8x_8)\,x_6-\tfrac{1}{2}x_7^2-\tfrac{1}{2}x_8^2.$$

Now we first distinguish two cases.

Case I. $c_6=0$. Then by Remark 3.1, $c_7=c_8=0$. Then by Remarks 5.2 and 5.3, $r_7=\frac{1}{2}=r_8$. If $r_5\neq 0$, take x_5 such that $0<|x_5|\leqslant\frac{1}{2}$, $x_2=x_4=0$, $x_7=\frac{1}{2}=x_8$ and choose x_6 such that $0<Q_8=x_5x_6-\frac{1}{4}\leqslant|x_5|\leqslant\frac{1}{2}<K$. If $r_5=0$, choose $x_2=x_4=0$, $x_5=x_6=1$, $x_7=\frac{1}{2}=x_8$. Then $Q_8=1-\frac{1}{4}=\frac{3}{4}<K$.

Case II. $c_6\neq 0$. Then by Remark 5.4, $c_6=\frac{1}{2}$. Here we distinguish a number of cases.

(A) $c_7=0=c_8$. Then by Remarks 5.2 and 5.3, $r_7=\frac{1}{2}=r_8$. Now take $x_2=x_4=0$, $x_7=\frac{1}{2}=x_8$, $x_5=r_5$ and $x_6=1$ or -1 such that $(x_5x_6)\geqslant 0$. Then

$$Q_8=|r_5|+\tfrac{1}{2}-\tfrac{1}{8}-\tfrac{1}{8}\qquad\text{so that}\quad \tfrac{1}{4}\leqslant Q_8\leqslant\tfrac{3}{4}<K.$$

(B) $c_7=\frac{1}{2}$, $c_8=0$. Then by Remark 5.7, $r_7=0$ and by Remark 5.3, $r_8=\frac{1}{2}$.

(i) If $0\leqslant|r_5|\leqslant\frac{1}{3}$, take $x_2=x_4=x_7=0$, $x_8=\frac{1}{2}$, $x_5=r_5$, and $x_6=1$ or -1 such that $(x_5x_6)\geqslant 0$. Then

$$Q_8=|r_5|+\tfrac{1}{2}-\tfrac{1}{8}\qquad\text{so that}\quad \tfrac{3}{8}\leqslant Q_8\leqslant\tfrac{17}{24}<K.$$

(ii) If $\frac{1}{3}<|r_5|\leqslant\frac{1}{2}$, take $x_2=x_4=0$, $x_8=\frac{3}{2}$, $x_5=r_5$, $x_6=2$ or -2 such that $(x_5x_6)<0$ and $x_7=1$ or -1 so that $(x_6x_7)>0$. Then

$$Q_8=-2\,|r_5|+2+1-\tfrac{1}{2}-\tfrac{9}{8}\qquad\text{so that}\quad \tfrac{3}{8}\leqslant Q_8\leqslant\tfrac{17}{24}<K.$$

Similarly by symmetry, we are through if $c_7=0$ and $c_8=\frac{1}{2}$.

(C) $c_7=\frac{1}{2}=c_8$ and $r_5\neq\frac{1}{2}$. Then by Remarks 5.2 and 5.3, $r_7=0=r_8$. Also $r_5\neq\frac{1}{2}$ so $-\frac{1}{2}<r_5<\frac{1}{2}$. Choose $x_2=x_4=x_7=x_8=0$, $x_5=r_5$ and $x_6=1$ or -1 such that $(x_5x_6)<0$. Then

$$Q_8=-|r_5|+\tfrac{1}{2}\qquad\text{so that}\quad 0<Q_8\leqslant\tfrac{1}{2}<K.$$

(D) $c_7=\frac{1}{2}=c_8$, $r_5=\frac{1}{2}$ but $b_4=0$. As in (C), $r_7=r_8=0$. As $b_4=0$, by Remark 3.1, $b_5=b_6=b_7=b_8=0$. We take $x_3\equiv r_3 \pmod 1$ such that $0<x_3\leqslant 1$.

(i) For $0<x_3\leqslant 0.8$, take $x_4=1$, $x_2=x_6=x_7=x_8=0$. Then

$$Q_8=x_3 \quad \text{so that} \quad 0<Q_8\leqslant 0.8<K.$$

(ii) For $0.8<x_3\leqslant 1$, take $x_4=x_8=1$, $x_2=x_6=x_7=0$. Then

$$Q_8=x_3-\tfrac{1}{2} \quad \text{so that} \quad 0.3\leqslant Q_8\leqslant \tfrac{1}{2}<K.$$

(E) $c_7=\frac{1}{2}=c_8$, $r_5=\frac{1}{2}$, $b_4\neq 0$ but $b_5\neq 0$.
As before $r_7=0=r_8$. By Remarks 5.4 and 5.5, $b_4=\frac{1}{2}=b_5$.

(i) If $0\leqslant r_3\leqslant\frac{1}{2}$, take $x_3=r_3$, $x_4=1$, $x_5=-\frac{1}{2}$ and $x_2=x_6=x_7=x_8=0$. Then

$$Q_8=r_3+\tfrac{1}{2}-\tfrac{1}{4} \quad \text{so that} \quad \tfrac{1}{4}\leqslant Q_8\leqslant\tfrac{3}{4}<K.$$

(ii) For $-\frac{1}{2}<r_3<0$, take $x_3=r_3$, $x_4=-1$, $x_5=\frac{1}{2}$ and $x_2=x_6=x_7=x_8=0$. Then

$$Q_8=-r_3+\tfrac{1}{2}-\tfrac{1}{4} \quad \text{so that} \quad \tfrac{1}{4}<Q_8\leqslant\tfrac{3}{4}<K.$$

(F) $c_7=\frac{1}{2}=c_8$, $r_5=\frac{1}{2}$, $b_4\neq 0$ but $b_5=0$. As before we have $r_7=r_8=0$, $b_4=\frac{1}{2}$. We divide this into a number of subcases.

(i) If $-\frac{1}{2}<r_3<\frac{1}{2}$, take $x_3=r_3$, $x_4=1$ or -1 such that $(x_3x_4)\leqslant 0$ and $x_2=x_6=x_7=x_8=0$. Then

$$Q_8=-|r_3|+\tfrac{1}{2} \quad \text{so that} \quad 0<Q_8\leqslant\tfrac{1}{2}<K.$$

(ii) $r_3=\frac{1}{2}$ but $b_6\neq 0$. Then either $-\frac{1}{2}<b_6<0$ or $0<b_6\leqslant\frac{1}{2}$. Take $x_2=x_7=x_8=0$, $x_6=1$, $x_5=-\frac{1}{2}$, $x_7=x_8=0$, $x_4=1$ or -1 such that $(b_6x_4)>0$, and $x_3=\frac{1}{2}$ or $-\frac{1}{2}$ such that $(x_3x_4)=-\frac{1}{2}$. Then

$$Q_8=-\tfrac{1}{2}+\tfrac{1}{2}+|b_6|-\tfrac{1}{2}+\tfrac{1}{2}=|b_6| \quad \text{so that} \quad 0<Q_8\leqslant\tfrac{1}{2}<K.$$

(iii) $r_3=\frac{1}{2}$ and $b_6=0$.

(a) If $-\frac{1}{2}<b_7\leqslant 0$, take $x_3=\frac{1}{2}$, $x_4=x_7=1$, $x_2=x_6=x_8=0$. Then

$$Q_8=\tfrac{1}{2}+\tfrac{1}{2}+b_7-\tfrac{1}{2}=\tfrac{1}{2}+b_7 \quad \text{so that} \quad 0<Q_8\leqslant\tfrac{1}{2}<K.$$

(b) If $0<b_7\leqslant\frac{1}{2}$, take $x_3=-\frac{1}{2}$, $x_4=x_6=x_7=1$, $x_5=-\frac{1}{2}$, $x_2=x_8=0$. Then

$$Q_8=-\tfrac{1}{2}+\tfrac{1}{2}+b_7-\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2}=b_7 \quad \text{so that} \quad 0<Q_8\leqslant\tfrac{1}{2}<K.$$

This proves Lemma 5.6 completely.

LEMMA 5.7. *Theorem* A *is true for* $\gamma = 1$.

Here $K^8 = \frac{2}{3}$ so that $K > 0.94$.
To prove Theorem A, we distinguish the following cases.

Case I. $c_6 = 0$. Then by Remark 3.1, $c_7 = c_8 = 0$. Also by Remark 5.7, $r_7 = \frac{1}{2}$ and by Remark 5.8, $r_8 = 0$ or $\frac{1}{2}$.

(i) If $r_5 \neq 0$, choose $x_2 = x_4 = 0$ $(x_7, x_8) \equiv (r_7, r_8) \pmod 1$ arbitrarily, $x_5 = r_5$, and an integer x_6 such that

$$0 < Q_8 = x_5 x_6 + \mu \leqslant |x_5| \leqslant \tfrac{1}{2} < K.$$

(ii) If $r_5 = 0 = r_8$, take $x_2 = x_4 = 0$, $x_5 = x_6 = 1$, $x_7 = \frac{1}{2}$, and $x_8 = 0$. Then

$$Q_8 = 1 - \tfrac{1}{8} = \tfrac{7}{8} < K.$$

(iii) If $r_5 = 0$, $r_8 = \frac{1}{2}$, choose $x_2 = x_4 = 0$, $x_5 = x_6 = 1$, $x_7 = x_8 = \frac{1}{2}$. Then

$$Q_8 = 1 - \tfrac{1}{8} - \tfrac{1}{4} = \tfrac{5}{8} < K.$$

Case II. $c_6 \neq 0$. Then by Remark 5.4, $c_6 = \frac{1}{2}$.

Now we divide into various subcases.

(A) $c_7 = c_8 = 0$. Then as above $r_7 = \frac{1}{2}$ and $r_8 = 0$ or $\frac{1}{2}$. Now choose $x_2 = x_4 = 0$, $x_5 = r_5$, $x_6 = 1$ or -1 such that $(x_5 x_6) \geqslant 0$, $x_7 = \frac{1}{2}$, and $x_8 = r_8$. Then

$$Q_8 = |r_5| + \tfrac{1}{2} - \tfrac{1}{8} - r_8^2 \qquad \text{so that} \quad \tfrac{1}{8} \leqslant Q_8 \leqslant \tfrac{7}{8} < K.$$

(B) $c_7 = \frac{1}{2}$, $c_8 = 0$. Then by Remark 5.2, $r_7 = 0$ and by Remark 5.3, $r_8 = 0$ or $\frac{1}{2}$.

(i) If $r_8 = \frac{1}{2}$, take $x_2 = x_4 = 0$, $x_5 = r_5$, $x_6 = 1$ or -1 such that $(x_5 x_6) \geqslant 0$, $x_7 = 0$ and $x_8 = \frac{1}{2}$. Then

$$Q_8 = |r_5| + \tfrac{1}{2} - \tfrac{1}{4} \qquad \text{so that} \quad \tfrac{1}{4} \leqslant Q_8 \leqslant \tfrac{3}{4} < K.$$

(ii) If $r_8 = 0$ but $r_5 \neq \frac{1}{2}$, take $x_2 = x_4 = x_7 = x_8 = 0$, $x_5 = r_5$, and $x_6 = 1$ or -1 such that $(x_5 x_6) \leqslant 0$. Then

$$Q_8 = -|r_5| + \tfrac{1}{2} \qquad \text{so that} \quad 0 < Q_8 \leqslant \tfrac{1}{2} < K.$$

(iii) If $r_8 = 0$ but $r_5 = \frac{1}{2}$, take $x_2 = x_4 = 0$, $x_5 = -\frac{1}{2}$, $x_6 = 2$, and $x_7 = 1 = x_8$. Then

$$Q_8 = -1 + 2 + 1 - \tfrac{1}{2} - 1 = \tfrac{1}{2} < K.$$

(C) $c_7=0$, $c_8=\frac{1}{2}$. Then by Remark 5.2, $r_7=\frac{1}{2}$ and by Remark 5.3, $r_8=\frac{1}{4}$ or $-\frac{1}{4}$. Now take $x_2=x_4=0$, $x_5=r_5$, $x_7=\frac{1}{2}$, $x_8=r_8$, and $x_6=1$ or -1 such that $(x_5x_6)\geqslant 0$. Then

$$Q_8=|r_5|+\tfrac{1}{2}\pm\tfrac{1}{8}-\tfrac{3}{16} \qquad \text{so that} \quad \tfrac{3}{16}\leqslant Q_8\leqslant\tfrac{15}{16}<K.$$

(D) $c_7=c_8=\frac{1}{2}$. Then by Remark 5.2, $r_7=0$ and by Remark 5.8, $r_4=\frac{1}{4}$ or $-\frac{1}{4}$.

(i) If $r_8=\frac{1}{4}$ and $-\frac{1}{2}<r_5\leqslant\frac{5}{16}$, take $x_2=x_4=0$, $x_5=r_5$, $x_6=1$, $x_7=0$, and $x_8=\frac{1}{4}$. Then

$$Q_8=r_5+\tfrac{1}{2}+\tfrac{1}{8}-\tfrac{1}{16} \qquad \text{so that} \quad \tfrac{1}{16}<Q_8\leqslant\tfrac{7}{8}<K$$

(ii) If $\frac{5}{16}<r_5\leqslant\frac{1}{2}$, take $x_5=1+r_5$, $x_6=2$, $x_7=-1$, $x_8=-\frac{7}{4}$, and $x_2=x_4=0$. Then

$$Q_8=2r_5-\tfrac{5}{16} \qquad \text{so that} \quad \tfrac{5}{16}<Q_8\leqslant\tfrac{11}{16}<K.$$

Similarly we can deal with $r_8=-\frac{1}{4}$. This completes the discussion when Q_4 is a zero form and the theorem is proved.

6. Proof of Theorem 1

We shall prove Theorem 1 by induction on n. We have already proved the theorem for $n=8$. So let $n\geqslant 10$ and assume the theorem is true for the quadratic form in $(n-2)$ variables. Let Q_n be a quadratic form of signature-2 having determinant $D\neq 0$. Set $K^n=2^n/3\cdot|D|$. Then witout loss of generality, we can take

$$Q_n=(x_1+a_2x_2+\cdots+a_nx_n)\,x_2+Q_{n-2}(x_3,\dots,x_n),$$

where Q_{n-2} is a quadratic form in $(n-2)$ variables and again is of signature-2. Let D' be the determinant of Q_{n-2}. Clearly $D'=4D$. Now we distinguish the following cases.

Case I. $K\geqslant 1$. Then the theorem is true by Remark 3.2.

Case II. $K<1$ and $r_2=0$. Then take $x_2=0$ and by the induction hypothesis, choose $(x_3,\dots,x_n)\equiv(r_3,\dots,r_n)\pmod 1$ such that

$$0<Q_n(x_1,\dots,x_n)=Q_{n-2}(x_3,\dots,x_n)\leqslant\left(\frac{2^{n-2}}{3}\cdot|D'|\right)^{1/(n-2)}$$

$$=\left(\frac{2^n}{3}\cdot|D|\right)^{1/(n-2)}=K^{n/(n-2)}<K \qquad (\because K<1).$$

Case III. $\frac{1}{2} \leqslant K < 1$ and $r_2 \not\equiv 0 \pmod 1$. Then the theorem is true by Remark 3.5.

Case IV. $K < \frac{1}{2}$ and $r_2 \not\equiv 0 \pmod 1$.

Now following the procedure of Birch [5], without loss of generality we can take

$$\begin{aligned} Q_n = {} & (x_1 + a_2 x_2 + \cdots + a_n x_n) x_2 + m_2(x_3 + b_4 x_4 + \cdots + b_n x_n) x_4 + \cdots \\ & + m_{(n-4)/2}(x_{n-5} + c_{n-4} x_{n-4} + \cdots + c_n x_n) x_{n-4} \\ & + Q_4(x_{n-3}, x_{n-2}, x_{n-1}, x_n), \end{aligned} \tag{6.1}$$

where Q_4 is a non-zero form of determinant $\Delta \neq 0$ and is of the type $(1, 3)$ or

$$\begin{aligned} Q_n = {} & (x_1 + a_2 x_2 + \cdots + a_n x_n) x_2 + m_2(x_3 + b_4 x_4 + \cdots + b_n x_n) x_4 + \cdots \\ & + m_{(n-2)/2}(x_{n-3} + d_{n-2} x_{n-2} + \cdots + d_n x_n) x_{n-2} - Q_2(x_{n-1}, x_n), \end{aligned} \tag{6.2}$$

where $Q_2(x_{n-1}, x_n)$ is a positive definite quadratic form of determinant $\Delta \neq 0$.

Now we distinguish these two cases.

(i) Let Q_n be in the form (6.1). Then by Lemma 2.4, $-Q_4(x_{n-3}, ..., x_n)$ and hence $-Q_n$ represents a number "a" such that

$$0 < a^4 \leqslant \tfrac{4}{7} |\Delta|.$$

Now

$$\begin{aligned} K^n &= \frac{2^n}{3} \cdot |D| = \frac{2^n}{3} \cdot \frac{1}{4} \cdot \frac{m_2^2}{4} \cdots \frac{m_{(n-4)/2}^2}{4} |\Delta| \\ &\geqslant \frac{2^n}{3} \cdot \underbrace{\frac{1}{4} \cdot \frac{1}{4} \cdots \frac{1}{4}}_{[(n-4)/2 \text{ times}]} \cdot \frac{7}{4} a^4 = \frac{28}{3} a^4 \qquad \therefore 0 < a^4 < \frac{3}{28} K^n \\ &= \frac{3}{28} \cdot K^{n-4} \cdot K^4 < \frac{K^4}{16} \qquad \left(\text{since } K < \frac{1}{2}, n \geqslant 10\right), \end{aligned}$$

and so $0 < |a| < K/2$.

(ii) Let Q_n be in the form (6.2). Then by Lemma 2.3, $Q_2(x_{n-1}, x_n)$ and hence $-Q_n$ represents a number "a" such that

$$0 < a^2 \leqslant \frac{4}{3} |\Delta|.$$

$$K^n = \frac{2^n}{3} \cdot |D| = \frac{2^n}{3} \cdot \frac{1}{4} \cdot \frac{m_2^2}{4} \cdots \frac{m_{(n-2)/2}^2}{4} \cdot |\Delta|$$

$$\geqslant \underbrace{\frac{2^n}{3} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdots \frac{1}{4}}_{[(n-2)/2 \text{ times}]} \cdot \frac{3}{4} a^2 = a^2. \qquad \text{So} \quad a^2 \leqslant K^n = K^2 \cdot K^{n-2}$$

$$< \frac{K^2}{4} \qquad \therefore 0 < |a| < \frac{K}{2} \qquad \left(\text{since } K < \frac{1}{2}\right).$$

Then in both cases (i) and (ii), the theorem follows by Lemma 2.6. We note that the results of Theorem 1 are the best possible as is shown by the following example.

Take

$$Q_n = x_1 x_2 + \cdots + x_{n-3} x_{n-2} - (x_{n-1}^2 + x_{n-1} x_n + x_n^2)$$

and

$$(r_1, ..., r_n) = (0, ..., 0).$$

Now

$$K^n = \frac{2^n}{3} \cdot |D| = \underbrace{\frac{2^n}{3} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdots \frac{1}{4}}_{[(n-2)/2 \text{ times}]} \cdot \frac{3}{4} = 1.$$

Obviously, the least positive value attained by Q_n at any integral value of $x_1, ..., x_n$ is 1. This completes the proof of Theorem 1.

7. Non-zero Forms

To prove the result for non-zero forms we need a few lemmas.

Lemma 7.1. *Let* $\phi(x_1, ..., x_4)$ *be a non-zero form of type* (2, 2) *and* det $D \neq 0$. *Let* $\beta_2 > \beta_1 \geqslant 0$ *be real numbers such that*

$$\frac{\beta_1}{\beta_2} \leqslant \frac{1}{4} \qquad \text{and} \qquad \beta_2 - \beta_1 > \left(\frac{1024}{81}\right)^{1/4} (|D|)^{1/4}.$$

Then given any real numbers $r_1, ..., r_4$ *there exist* $(x_1, ..., x_4) \equiv (r_1, ..., r_4)$ (mod 1) *satisfying*

$$\beta_1 < \phi(x_1, x_2, x_3, x_4) < \beta_2.$$

For proof see Theorem 3 of Bambah, Dumir, and Hans-Gill [3].

LEMMA 7.2. *If* $Q_5 = Q_5(x_1, ..., x_5)$ *is a non-zero form of type* (2, 3) *and* $\det D \neq 0$, *and* $0 < t \leqslant \frac{1}{4}$ *and* $f(t) \geqslant 16/(1-t)^5$, *then*

$$t(f(t)\,|D|)^{1/5} < Q_5 < (f(t)\,|D|)^{1/5} = d \tag{7.1}$$

is always solvable.

Proof. Applying the reduction procedure of Bambah, Dumir, and Hans-Gill [3], we can take

$$Q_5 = -(x_1 + h_2 x_2 + \cdots + h_5 x_5)^2 + Q_4'(x_2, ..., x_5),$$

where $Q_4'(x_2, ..., x_5)$ is a non-zero quadratic form of type (2, 2) and det D with $|D| > 2$ (see conditions (4.2) and (4.3) of Theorem 1 in [3]). Now (7.1) will be solvable if

$$0 < Q_4' - (x_1 + h_2 x_2, ...)^2 - td < (1-t)d = \delta \tag{7.2}$$

is solvable. But

$$f(t) \geqslant \frac{16}{(1-t)^5} \Rightarrow (1-t)^5 f(t)\,|D|$$
$$> 32 \Rightarrow \delta > 2 \qquad \text{where} \quad \delta = (1-t)d.$$

Let m be an integer defined by

$$m < \delta \leqslant m + 1.$$

Then $m \geqslant 2$. By Lemma 2.1(b), (7.2) is solvable if

$$\frac{1}{4} < Q_4' - td < \frac{m^2}{4} + \delta$$

or if

$$\beta_1 = \frac{1}{4} + td < Q_4' < \frac{m^2}{4} + d = \beta_2$$

is solvable.

Now

$$\frac{\beta_1}{\beta_2} = \frac{1/4 + td}{m^2/4 + d} < \frac{1}{4} \qquad \left(\because m \geqslant 2 \text{ and } t \leqslant \frac{1}{4}\right).$$

By Lemma 7.1, the lemma is true if

$$\frac{m^2 - 1}{4} + \delta > \left(\frac{1024}{81}\,|D|\right)^{1/4} = \left(\frac{1024}{81} \cdot \frac{d^5}{f(t)}\right)^{1/4}. \tag{7.3}$$

But $f(t) \geqslant 16/(1-t)^5$ and so $1/f(t) \leqslant (1-t)^5/16$. Then (7.3) holds if

$$\frac{m^2-1}{4}+\delta > \left(\frac{1024}{81}\cdot\frac{d^5(1-t)^5}{16}\right)^{1/4} = \left(\frac{64}{81}\delta^5\right)^{1/4},$$

or if $((m^2-1)/4+\delta)\,\delta^{-5/4} > (\frac{64}{81})^{1/4}$. But $g(\delta) = ((m^2-1)/4+\delta)\,\delta^{-5/4}$ is a decreasing function of δ and $\delta \leqslant m+1$

$$\therefore g(\delta) \geqslant g(m+1) = \frac{(m+3)(m+1)^{-1/4}}{4} \geqslant \frac{5}{4}(3)^{-1/4} > \left(\frac{64}{81}\right)^{1/4} \qquad (m \geqslant 2).$$

This proves the lemma.

LEMMA 7.3. *If* $Q_6 = Q_6(x_1, ..., x_6)$ *is a non-zero quadratic form of type* $(2, 4)$ *and* $\det D \neq 0$, *then*

$$0 < Q_6 < (k\,|D|)^{1/6} \text{ is solvable for all } k \geqslant 32. \tag{7.4}$$

Proof. From the homogeneous results for quadratic forms of type (2, 4) (see Watson [17]), we know that there exists a negative value $-a$ attained by Q_6 such that

$$0 < a \leqslant (\tfrac{64}{3}\,|D|)^{1/6}.$$

Using unimodular transformation, if necessary, we can assume $Q_6 = -a(x_1+h_2x_2+\cdots)^2 + Q_5(x_2, ..., x_6)$ where $Q_5(x_2, ..., x_6)$ is a quadratic form of type (2, 3). Also without loss of generality we may assume $a = 1$, so that $|D| > \frac{3}{64}$ and $k\,|D| \geqslant \frac{3}{64}\cdot 32 = \frac{3}{2} > 1$. Now (7.4) is solvable if

$$0 < -(x_1+h_2x_2, ...)^2 + Q_5(x_2, ..., x_6) < (k\,|D|)^{1/6} = d \tag{7.5}$$

is solvable.

Using Lemma 2.1(b), (7.5) is solvable if

$$\frac{1}{4} < Q_5 < \frac{n^2}{4} + d \text{ is solvable,} \tag{7.6}$$

where n is the integer defined by $n < d \leqslant n+1$. Here also $t = (1/4)/(n^2/4+d) < \frac{1}{4}$ (since $d \geqslant 1$). Using Lemma 7.2, (7.6) is solvable, if

$$\frac{n^2-1}{4}+d \geqslant (16\,|D|)^{1/5} = \left(\frac{16d^6}{k}\right)^{1/5},$$

or if

$$\left(\frac{n^2-1}{4}+d\right)d^{-6/5} > \left(\frac{16}{k}\right)^{1/5}. \tag{7.7}$$

But $g(d) = ((n^2-1)/4 + d)\, d^{-6/5}$ is a decreasing function of d and $d \leqslant n+1$, so $g(d) \geqslant g(n+1) = (n+3)/(4(n+1)^{1/5})$. Since $n \geqslant 1$, $g(n+1) \geqslant 1/2^{1/5} \geqslant (16/k)^{1/5}$ if $k \geqslant 32$. So (7.7) holds good. This proves the lemma.

LEMMA 7.4. *Let $Q(x, y)$ be an indefinite quadratic form of* det $-d$ *$(d>0)$. Then given any real numbers x_0, y_0 and $\mu>0$, there exist $(x, y) \equiv (x_0, y_0)$* (mod 1) *such that*

$$0 < Q(x, y) + \mu \leqslant \max(2\sqrt{d}, \sqrt{d + 4\mu\sqrt{d}} - \sqrt{d}).$$

For the proof see Lemma 5′ in [1].

LEMMA 7.5. *Let $Q_n = Q_n(x_1, ..., x_n)$ be an indefinite quadratic form in $n \geqslant 8$ variables. Let Q_n be of signature* -2, det $D \neq 0$, *and $m(Q_n) > 0$. Then*

$$Q_n \sim \Psi(x_1 + a_{12}x_2 + \cdots x_2 + a_{23}x_3 + \cdots) + Q_{n-2}(x_3, ..., x_n),$$

where Q_{n-2} is again of signature -2 and $\Psi(x, y)$ is an indefinite binary quadratic form of det $-d$ $(d>0)$. *Further*

$$d^{n/2} \leqslant (\tfrac{16}{9})^{1/3}\, |D|\, (\tfrac{5}{6})^{(n-4)(n+2)/4}.$$

This follows from Lemma 13 *in* [2].

Remark 7.1. Let $Q_n = Q_n(x_1, ..., x_n)$ be any indefinite quadratic form with det $D \neq 0$ and $m(Q_n) = 0$, then by a result of Watson [15], (1.1) is satisfied for all $\Gamma > 0$. So, in the rest of the paper, we shall assume $m(Q_n) > 0$.

Theorem 2′. Let $Q_8 = Q_8(x_1, ..., x_8)$ be an indefinite non-zero quadratic form of type (3, 5) with $m(Q_8) > 0$ and det $D \neq 0$. Then

$$0 < Q_8(x_1, ..., x_8) < (116\, |D|)^{1/8}$$

is solvable.

Proof. By Lemma 7.5 we can take

$$Q_8 = \Psi(x_1 + a_{12}x_2 + \cdots x_2 + a_{23}x_3 + \cdots) + Q_6(x_3, ..., x_8),$$

where Q_6 is of type (2, 4) and det $-D/d$ with

$$d^4 < (\tfrac{16}{9})^{1/3}\, |D|\, (\tfrac{5}{6})^{10} < 0.1958\, |D|. \tag{7.9}$$

By Lemma 7.3 we can choose

$$(x_3, ..., x_8) \equiv (r_3, ..., r_8) \pmod 1$$

satisfying

$$0 < Q_6(x_3, ..., x_8) = \mu < \left[(32.016)\frac{|D|}{d}\right]^{1/6}.$$

Take $y_0 = r_2 + a_{23}x_2 + \cdots$ and $x_0 + a_{12}y_0 = r_1 + a_{12}r_2 + a_{13}x_3 + \cdots$ so that $Q_8 = Q(x, y) + \mu$ where

$$Q(x, y) = \Psi(x_1 + a_{12}x_2, x_2).$$

By Lemma 7.4, there exists $(x, y) \equiv (x_0, y_0) \pmod 1$ and hence $(x_1, x_2) \equiv (r_1, r_2) \pmod 1$ satisfying

$$0 < Q_8 \leqslant \max(2\sqrt{d}, \sqrt{d + 4\mu\sqrt{d}} - \sqrt{d}).$$

But by (7.9)

$$2\sqrt{d} < 2(0.1958)^{1/8}\,|D|^{1/8} < (116\,|D|)^{1/8},$$

so the lemma will be true if

$$\sqrt{d + 4\mu\sqrt{d}} - \sqrt{d} < (116\,|D|)^{1/8} = 2\left(\frac{29}{64}|D|\right)^{1/8},$$

or if

$$\mu\sqrt{d} < \left(\frac{29}{64}|D|\right)^{1/4} + \left(\frac{29}{64}|D|\right)^{1/8}\sqrt{d},$$

or if

$$\left(32.016\frac{|D|}{d}\right)^{1/6}\sqrt{d} < \left(\frac{29}{64}|D|\right)^{1/4} + \left(\frac{29}{64}|D|\right)^{1/8}\sqrt{d},$$

or if

$$\left(32.016\frac{|D|}{d^4}\right)^{1/6} < \left(\frac{29}{64}\frac{|D|}{d^4}\right)^{1/4} + \left(\frac{29}{64}\frac{|D|}{d^4}\right)^{1/8},$$

or if

$$(70.656\,t^8)^{1/6} < t^2 + t \qquad \text{where} \quad t^8 = \frac{29}{64}\frac{|D|}{d^4},$$

or if

$$(70.656\,t^2) < (t+1)^6,$$

or if

$$f(t) = (t+1)^6 - 70.656\,t^2 > 0.$$

Now $f(t)$ is an increasing function of t for $t>1$ and

$$t^8 = \frac{29}{64}\frac{|D|}{d^4} > \frac{29}{64}\frac{1}{0.1958} > 1.1105$$

$\therefore f(t) > f(1.1105) > 0$ which proves (7.11) and hence the Theorem.

Remark. Combining Theorems 1 and 2 we get Theorem 3.

First we prove Theorem 4 for $n=10$.

LEMMA 7.6. *Let $Q_{10} = Q_{10}(x_1, ..., x_{10})$ be a non-zero quadratic form of type* (4, 6) *and* $\det D \neq 0$, *then*

$$0 < Q_{10} < K \qquad \textit{where} \quad K^{10} = 2^{10}\frac{|D|}{3} \tag{7.12}$$

is solvable.

Proof. Proceeding as in the last lemma we can write $Q_{10} = Q(x, y) + \mu$ where

$$Q(x, y) = \Psi(x_1 + a_{12}x_2, x_2) \qquad \text{and} \qquad 0 < Q_8(x_3, ..., x_8) = \mu < \left(116\frac{|D|}{d}\right)^{1/8}.$$

Also $\det Q(x, y) = -d$ $(d>0)$ such that

$$d^5 < (\tfrac{16}{9})^{1/3}\,|D|\,(\tfrac{5}{6})^{1/18} < 0.0456\,|D|. \tag{7.13}$$

Now $0 < Q_{10} = Q(x, y) + \mu < K$ is solvable as in the last lemma if

$$\sqrt{d + 4\mu\sqrt{d}} - \sqrt{d} < 2(\tfrac{1}{3}\,|D|)^{1/10},$$

or if

$$\mu\sqrt{d} < (\tfrac{1}{3}\,|D|)^{1/5} + (\tfrac{1}{3}\,|D|)^{1/10}\sqrt{d},$$

or if

$$\left(\frac{116\,|D|}{d^5}\right)^{1/8} < \left(\frac{1}{3}\frac{|D|}{d^5}\right)^{1/5} + \left(\frac{1}{3}\frac{|D|}{d^5}\right)^{1/10},$$

or if

$$(348t^{10})^{1/8} < t^2 + t, \tag{7.14}$$

where $t^{10} = \frac{1}{3}\cdot|D|/d^5$. Then

$$t > \left(\frac{1}{0.1368}\right)^{1/10} = (7.3)^{1/10} \Rightarrow t > 1.2.$$

So (7.14) holds if $(348t^2)^{1/8} < t+1$, or if $f(t) = (t+1)^8 - 348t^2 > 0$. Now $f(t)$ is an increasing function for $t > 1 \Rightarrow f(t) > f(1.2) > 0$.

Thus the lemma is proved.

Now we prove Theorem 4 for $n \geqslant 10$.

LEMMA 7.7. *Let $Q_n = Q_n(x_1, ..., x_n)$ be an indefinite quadratic non-zero form of signature -2. Then*

$$0 < Q_n(x_1, ..., x_n) < \left(\frac{2^n}{3}|D|\right)^{1/n}$$

is solvable for $n \geqslant 10$.

Proof. We shall prove the lemma by induction on n. We have already proved it for $n = 10$. So, we assume the lemma is true for $n-2$ $(n \geqslant 12)$;

$$0 < Q_{n-2}(x_3, ..., x_n) = \mu < 2\left(\frac{|D|}{3d}\right)^{1/(n-2)}$$

is solvable where

$$Q_n = Q(x, y) + Q_{n-2}.$$

Here $Q(x, y)$ is the indefinite binary quadratic form of det $-d$ $(d > 0)$. But Lemma 7.5 gives $d^{n/2} < \frac{1}{3}|D|$ for $n \geqslant 12$. So the lemma will be true if

$$\frac{\mu}{\sqrt{d}} < \left(\frac{|D|}{3d^{n/2}}\right)^{2/n} + \left(\frac{|D|}{3d^{n/2}}\right)^{1/n},$$

or if

$$2\left(\frac{|D|}{3d^{n/2}}\right)^{1/(n-2)} < \left(\frac{|D|}{3d^{n/2}}\right)^{2/n} + \left(\frac{|D|}{3d^{n/2}}\right)^{1/n},$$

or if

$$(2(t^n)^{1/(n-2)}) < t^2 + t \qquad \text{where} \quad t^n = \frac{|D|}{3d^{n/2}} > 1,$$

or if

$$2(t^2)^{1/(n-2)} < t+1,$$

or if

$$t+1 > 2\sqrt{t} \qquad \left(\because \frac{2}{n-2} < \frac{1}{2}\right)$$

or if

$$(\sqrt{t}-1)^2 > 0, \text{ which is always true.}$$

This completes the induction, and hence the lemma is true for all n. Combining Theorem 1 and this lemma we get Theorem 4.

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